Exercice 4. 3

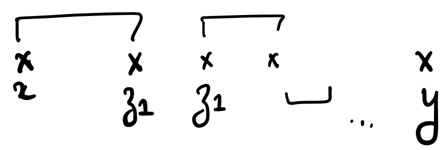
- $\mathcal{L}_I = -\frac{1}{2}\mu^2\phi^2$ Compute $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \langle 0|T\{\phi(x)\phi(y)\,e^{iS_I}\}|0\rangle\Big|_{\text{connected}}$$

$$= \langle 0|\sum_{n=0}^{+\infty}\left(-\frac{i}{2}\mu^2\right)^n\frac{1}{n!}\,T\left\{\phi(x)\phi(y)\int d^4z_1\ldots\int d^4z_n\,\phi^2(z_1)\ldots\phi^2(z_n)\right\}|0\rangle_{c\ldots}$$

$$= \sum_{n=0}^{+\infty}\left(-\frac{i}{2}\mu^2\right)^n\frac{1}{n!}\langle 0|T\left\{\phi(x)\phi(y)\int d^4z_1\ldots d^4z_n\,\phi^2(z_1)\ldots\phi^2(z_n)\right\}|0\rangle_{\text{connected}}$$

$$\phi(x)\phi(y)\ \phi(z_1)\cdots\phi(z_n)$$



$2^n n!$ ways to connect all of them in a "line".

$$= \sum_{n=0}^{+\infty}\left(-\frac{i}{2}\mu^2\right)^n\frac{1}{n!}2^n n!\int D_F(x,z_1)D_F(z_1,z_2)\ldots D_F(z_{n-1},z_n)D_F(z_n,y)\,d^4z_1\ldots d^4z_n$$

$$= \sum_{n=0}^{+\infty}\left(-i\mu^2\right)^n\int\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}e^{ik(x-z_1)}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{ip(z_1-z_2)}\ldots d\ldots$$

$$= \sum_{n=0}^{+\infty}\left(-i\mu^2\right)^n\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\varepsilon}\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{ikx}e^{-ikz_2}(2\pi)^4\delta(k-p)\ldots d^4z\ldots$$

$$= \sum_{n=0}^{+\infty}\left(-i\mu^2\right)^n\int\frac{d^4k}{(2\pi)^4}\left(\frac{i}{k^2-m^2+i\varepsilon}\right)^2\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}e^{ip(z_2-z_3)}\ldots$$

$$= \sum_{n=0}^{+\infty}\left(-i\mu^2\right)^n\int\frac{d^4k}{(2\pi)^4}\left(\frac{i}{k^2-m^2+i\varepsilon}\right)^{n+1}e^{-ik(x-y)}$$

$$= \int\frac{d^4k}{(2\pi)^4}\sum_{n=0}^{+\infty}\left[\frac{\mu^2}{k^2-m^2+i\varepsilon}\right]^n\frac{i}{k^2-m^2+i\varepsilon}e^{-ik(x-y)}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{1 - \frac{\mu^2}{k^2 - m^2 + i\varepsilon}} \frac{i}{k^2 - m^2 + i\varepsilon} e^{-ik(x-y)}$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - (m^2 + \mu^2) + i\varepsilon} e^{-ik(x-y)}$$

! Ok, what we needed to prove indeed.

## Exercise 5.1

a. $\mathcal{L}_I = -\frac{1}{4!} \lambda \phi^4 \qquad \phi\phi \to \phi\phi$

To compute $\mathcal{V}$ we need to evaluate $\langle p_1, p_2 | \hat{S} - \mathbb{1} | k_1, k_2 \rangle$

$$\rightsquigarrow \langle p_1, p_2 | T\{e^{iS_I}\} - \mathbb{1} | k_1, k_2 \rangle \Big|_{\substack{\text{connected} \\ \text{amputated}}} \qquad S_I = \int d^4z \, \mathcal{L}_I$$

$$\langle p_1, p_2 | T\{e^{iS_I}\} - \mathbb{1} | k_1, k_2 \rangle \Big|_{c.a} = -i \frac{\lambda}{4!} \int d^4z \, \langle p_1, p_2 | \phi^4(z) | k_1, k_2 \rangle + \mathcal{O}(\lambda^2)$$

$$\langle p_1, p_2 | \phi^4(z) | k_1, k_2 \rangle = \langle p_1, p_2 | \phi(z)\phi(z)\phi(z)\phi(z) | k_1, k_2 \rangle \qquad 4! \text{ ways}$$

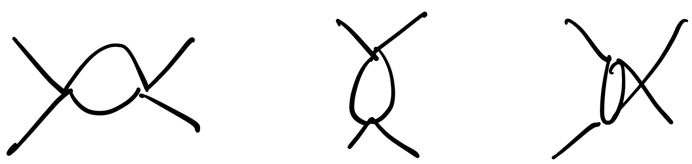
4 2 3 1

$$\langle p_1, p_2 | T\{e^{iS_I}\} - \mathbb{1} | k_1, k_2 \rangle \Big|_{c.a} = -i\lambda \int d^4z \, e^{iz(p_1 + p_2 - k_1 - k_2)} + \mathcal{O}(\lambda^2)$$

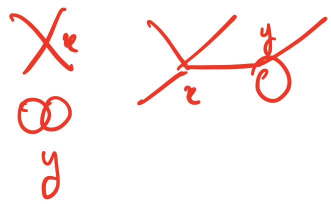
$$= -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) + \mathcal{O}(\lambda^2)$$

$$i\mathcal{V} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) = -i\lambda (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) + \mathcal{O}(\lambda^2)$$

$$\rightsquigarrow \boxed{\mathcal{V} = -\lambda + \mathcal{O}(\lambda^2)}$$

b. 

Do not contribute:



## Exercice 5.3

$$\mathcal{L}_I = -\frac{1}{3!} g \phi^3 \qquad \phi\phi \to \phi\phi$$

a) Diagrammes à l'ordre le plus bas en $g$ :

$$\langle p_1, p_2 | T\{e^{iS_I}\} - 1 | k_1, k_2 \rangle_{c.a} = -\frac{1}{3!} ig \int d^4z \, \langle p_1, p_2 | T\{\phi^3(z)\} | k_1, k_2 \rangle_{c.a}$$

$$+ \left(-\frac{1}{3!} ig\right)^2 \frac{1}{2} \int d^4z_1 \int d^4z_2 \, \langle p_1, p_2 | T\{\phi^3(z_1)\phi^3(z_2)\} | k_1, k_2 \rangle_{c.a}$$

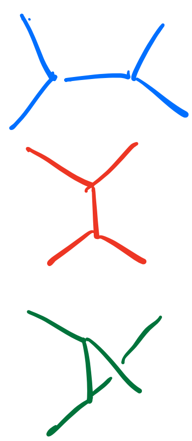
$$+ \mathcal{O}(g^3)$$

Can we do something with the first one : NO !

$$\langle p_1, p_2 | T\{\phi^3(z)\} | k_1, k_2 \rangle = \langle p_1, p_2 | T\{\phi(z)\phi(z)\phi(z)\} | k_1, k_2 \rangle$$

Lowest order : $g^2$

$$\langle p_1, p_2 | T\{\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_2)\phi(z_2)\phi(z_2)\} | k_1, k_2 \rangle :$$



Therefore we can decompose the total amplitude into 3 contributions:

$$\langle p_1, p_2 | \hat{T} | k_1, k_2 \rangle = S + T + U + \mathcal{O}(g^3)$$

5496.70

with $$S = \frac{1}{2}\left(\frac{ig}{3!}\right)^2 (3!)^2\, 2 \int d^4 z_1 \int d^4 z_2\, e^{i(p_1+p_2)z_1} D_F(z_1 - z_2)\, e^{-i(k_1+k_2)z_2}$$

$$T = \frac{1}{2}\left(\frac{ig}{3!}\right)^2 (3!)^2\, 2 \int d^4 z_1 \int d^4 z_2\, e^{i(p_1-k_1)z_1} D_F(z_1 - z_2)\, e^{i(p_2-k_2)z_2}$$

$$U = \frac{1}{2}\left(\frac{ig}{3!}\right)^2 (3!)^2\, 2 \int d^4 z_1 \int d^4 z_2\, e^{i(p_1-k_2)z_1} D_F(z_1 - z_2)\, e^{i(p_2-k_1)z_2}$$

b) $$S = (ig)^2 \int d^4 z_1 \int d^4 z_2\, e^{i(p_1+p_2)z_1} \int \frac{d^4 q}{(2\pi)^4} \frac{i e^{-iq(z_1-z_2)}}{q^2 - m^2 + i\varepsilon}\, e^{i(k_2+k_1)z_2}$$

$$= (ig)^2 \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q)\, (2\pi)^4 \delta^{(4)}(k_2 + k_1 + q)$$

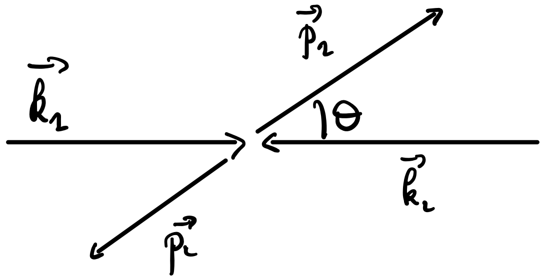
$$= (ig)^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \frac{i}{(k_1+k_2)^2 - m^2 + i\varepsilon}$$

$$T = (ig)^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \frac{i}{(k_1 - p_1)^2 - m^2 + i\varepsilon}$$

$$U = (ig)^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \frac{i}{(k_1 - p_2)^2 - m^2 + i\varepsilon}$$

Thus: $$\mathcal{M} = -g^2 \left\{ \frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{u - m^2} \right\}.$$

c)




Centre of mass frame: $\vec{k}_1 = -\vec{k}_2$

$k_1^\mu = (E, \vec{k}) \quad k_2^\mu = (E, -\vec{k})$

for particles with the same mass

$$E = \sqrt{m^2 + k^2} \qquad \vec{k} = \vec{k}_1 = -\vec{k}_2$$

$$s = (k_1 + k_2)^2 = 2m^2 + 2(E^2 + k^2) = 4m^2 + 4|\vec{k}|^2$$

$$\rightsquigarrow \boxed{\frac{s - 4m^2}{m^2} = 4\frac{|\vec{k}|^2}{m^2}}$$

UR-limit: $\varepsilon_{UR} = \dfrac{m^2}{s - 4m^2} = \dfrac{m^2}{4|\vec{k}|^2} \ll 1 \rightsquigarrow s = 4m^2 + \dfrac{m^2}{\varepsilon_{UR}}$

$$\rightsquigarrow s - m^2 = 3m^2 + \frac{m^2}{\varepsilon_{UR}} = m^2\left\{3 + \frac{1}{\varepsilon_{UR}}\right\} \simeq \frac{m^2}{\varepsilon_{UR}}\left(1 + \mathcal{O}(\varepsilon_{UR})\right)$$

$$\rightsquigarrow t = (k_1 - p_1)^2 = 2m^2 - 2(E^2 - k^2\cos\theta) = -2k^2(1 - \cos\theta) = -\frac{m^2}{2\varepsilon_{UR}}(1 - \cos\theta)$$

$$t - m^2 = -\frac{m^2}{2\varepsilon_{UR}}(1 - \cos\theta + 2\varepsilon_{UR})$$

$$\frac{1}{t - m^2} = -\frac{2\varepsilon_{UR}}{m^2}\frac{1}{1 - \cos\theta + 2\varepsilon_{UR}} \simeq -\frac{2\varepsilon_{UR}}{m^2}\frac{1}{1 - \cos\theta} + \mathcal{O}(\varepsilon_{UR}^2)$$

$$\rightsquigarrow \frac{1}{u - m^2} = -\frac{2\varepsilon_{UR}}{m^2}\frac{1}{1 + \cos\theta}$$

D'où $\sigma_{UR} \simeq \dfrac{g^2 \varepsilon_{UR}}{m^2}\left\{-1 + \dfrac{2}{1 - \cos\theta} + \dfrac{2}{1 + \cos\theta}\right\} = \dfrac{g^2}{m^2}\varepsilon_{UR}\dfrac{3 + \cos^2\theta}{\sin^2\theta}$

$$\boxed{\sigma_{UR} \simeq \frac{g^2}{s - 4m^2}\frac{3 + \cos^2\theta}{\sin^2\theta}}$$

NR - limit :

Same arguments yield : $\boxed{\mathcal{M}_{NR} = \frac{5}{3}\frac{g^2}{m^2}}$

d) $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E^2}|\mathcal{M}|^2$ with $d\Omega = \sin\theta \, d\theta \, d\varphi$

$\sigma = \frac{1}{2}\int \frac{d\sigma}{d\Omega} \, d\Omega$

↳ identical particles going out (undistinguishables)

NR: $\sigma_{NR} = \frac{1}{64\pi^2 E^2}\frac{g^4}{m^4}\frac{25}{9}4\pi$

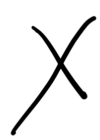
$= \frac{25}{144\pi m^4}\frac{1}{E^2}$

UR: $\sigma_{UR} = \frac{g^4}{(s-4m^2)^2}\frac{1}{64\pi^2 E^2} 2\pi \int_{-1}^{1} \frac{(3+\cos^2\theta)^2}{\sin^4\theta} \, d\cos\theta$

$= \frac{g^4}{32\pi E^2}\frac{1}{(s-4m^2)^2}\int_{-1}^{1}\frac{(3+x^2)^2}{(1-x^2)^2} \, dx$ ⇒ divergence.

The cross section diverges when the energy of the incoming particles increases!

e) We add $\mathcal{L}_I = -\frac{1}{4!}\frac{g^2}{3m^2}\phi^4$

At lowest order we add the diagram 

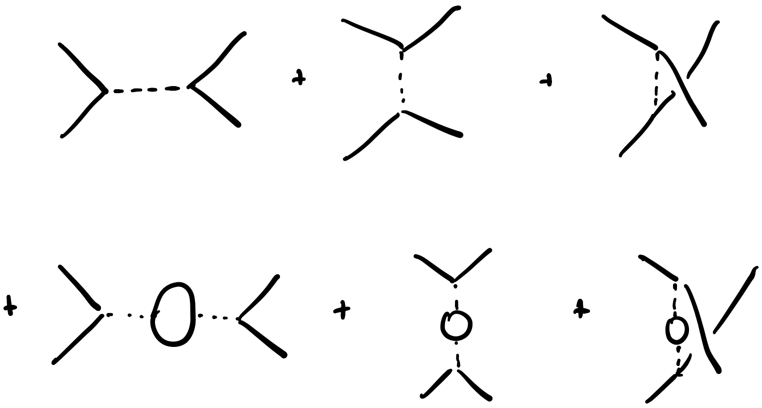
$\mathcal{M} = \mathcal{M}_{before} - \frac{g^2}{3m^2}$

# Exercice 5.2

$\mathcal{L}_I = \lambda \phi \psi^2 \quad \phi$ masse $m \quad \psi$ masse $M$

$\psi(k_1)\psi(k_2) \to \psi(p_1)\psi(p_2)$

a)



b) $\langle p_1, p_2 | \, i\hat{T} \, | k_1, k_2 \rangle_{c.a} = \frac{1}{2}(i\lambda)^2 \int d^4 z_1 \, d^4 z_2 \, \overbrace{\langle p_1, p_2 | T\{\phi(z_1)\psi(z_1)}^{\times 2}\psi(z_1)\phi(z_2)\psi(z_2)\psi(z_2)\} \ldots \rangle$

Exercice 6.1

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$$

$$\cdot\ [\gamma^\mu,[\gamma^\nu,\gamma^\rho]] = [\gamma^\mu, \{\gamma^\nu,\gamma^\rho\} - 2\gamma^\rho\gamma^\nu]$$
$$= -2[\gamma^\mu,\gamma^\rho\gamma^\nu]$$
$$= -2\left(\gamma^\rho[\gamma^\mu,\gamma^\nu] + [\gamma^\mu,\gamma^\rho]\gamma^\nu\right)$$
$$= -2\left(\gamma^\rho\{\gamma^\mu,\gamma^\nu\} - 2\gamma^\rho\gamma^\nu\gamma^\mu + \{\gamma^\mu,\gamma^\rho\}\gamma^\nu - 2\gamma^\rho\gamma^\mu\gamma^\nu\right)$$
$$= -4\left(\gamma^\rho\eta^{\mu\nu} + \gamma^\nu\eta^{\mu\rho} - \gamma^\rho\{\gamma^\nu,\gamma^\mu\}\right)$$
$$= -4\left(\gamma^\nu\eta^{\mu\rho} - \gamma^\rho\eta^{\mu\nu}\right)$$

$$[S^{\mu\nu},S^{\rho\sigma}] = -\frac{1}{16}\left[[\gamma^\mu,\gamma^\nu],[\gamma^\rho,\gamma^\sigma]\right]$$
$$= -\frac{1}{16}\left([\gamma^\mu\gamma^\nu,[\gamma^\rho,\gamma^\sigma]] - [\gamma^\nu\gamma^\mu,[\gamma^\rho,\gamma^\sigma]]\right)$$
$$= -\frac{1}{16}\Big(\gamma^\mu[\gamma^\nu,[\gamma^\rho,\gamma^\sigma]] + [\gamma^\mu,[\gamma^\rho,\gamma^\sigma]]\gamma^\nu$$
$$- \gamma^\nu[\gamma^\mu,[\gamma^\rho,\gamma^\sigma]] - [\gamma^\nu,[\gamma^\rho,\gamma^\sigma]]\gamma^\mu\Big)$$
$$= \frac{1}{4}\Big(\gamma^\mu(\gamma^\rho\eta^{\nu\sigma} - \gamma^\sigma\eta^{\nu\rho}) + (\gamma^\rho\eta^{\mu\sigma} - \gamma^\sigma\eta^{\mu\rho})\gamma^\nu$$
$$- \gamma^\nu(\gamma^\rho\eta^{\mu\sigma} - \gamma^\sigma\eta^{\mu\rho}) - (\gamma^\rho\eta^{\nu\sigma} - \gamma^\sigma\eta^{\nu\rho})\gamma^\mu\Big)$$
$$= \frac{1}{4}\left([\gamma^\mu,\gamma^\rho]\eta^{\nu\sigma} + [\gamma^\rho,\gamma^\nu]\eta^{\mu\sigma} + [\gamma^\nu,\gamma^\sigma]\eta^{\mu\rho} + [\gamma^\sigma,\gamma^\mu]\eta^{\nu\rho}\right)$$
$$= -i\left(S^{\mu\rho}\eta^{\nu\sigma} + S^{\rho\nu}\eta^{\mu\sigma} + S^{\nu\sigma}\eta^{\mu\rho} + S^{\sigma\mu}\eta^{\nu\rho}\right)$$
$$= -i\left(S^{\mu\rho}\eta^{\nu\sigma} + S^{\nu\sigma}\eta^{\mu\rho} - S^{\nu\rho}\eta^{\mu\sigma} - S^{\mu\sigma}\eta^{\nu\rho}\right)$$

## Exercice 6.2

$$\{\gamma^\mu, \gamma^\nu\} = 0 \text{ for } \mu \neq \nu$$
$$\{\gamma^i, \gamma^i\} = -2 \quad \{\gamma^0, \gamma^0\} = 2$$

$$\begin{aligned}(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= +\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\ &= -\gamma^2\gamma^3\gamma^2\gamma^3 \\ &= -\gamma^3\gamma^3 = 1\end{aligned}$$

$$\{\gamma^\mu, \gamma^5\} = i\left(\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu\right) = 0 \quad \text{(test all possible case)}$$

$$\begin{aligned}[S^{\mu\nu}, \gamma^5] &= \frac{i}{4}\left[[\gamma^\mu, \gamma^\nu], \gamma^5\right] = \frac{i}{4}\left([\gamma^\mu\gamma^\nu, \gamma^5] - [\gamma^\nu\gamma^\mu, \gamma^5]\right) \\ &= \frac{i}{4}\left(\gamma^\mu[\gamma^\nu, \gamma^5] + [\gamma^\mu, \gamma^5]\gamma^\nu - \gamma^\nu[\gamma^\mu, \gamma^5] - [\gamma^\nu, \gamma^5]\gamma^\mu\right) \\ &= \frac{i}{4}\left(-2\gamma^\mu\gamma^5\gamma^\nu - 2\gamma^5\gamma^\mu\gamma^\nu + 2\gamma^\nu\gamma^5\gamma^\mu + 2\gamma^5\gamma^\nu\gamma^\mu\right) \\ &= \frac{i}{2}\left(\gamma^5\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\nu - \gamma^5\gamma^\nu\gamma^\mu + \gamma^5\gamma^\nu\gamma^\mu\right) \\ &= 0\end{aligned}$$

## Exerice 6.4

a) $$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad (\sigma^\mu)^\dagger = \sigma^\mu \qquad (\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu$$

$$(\gamma^\mu)^\dagger = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & \end{pmatrix}$$

$$\gamma^0\gamma^\mu\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma^\mu & 0 \\ 0 & \bar\sigma^\mu \end{pmatrix} = \begin{pmatrix} 0 & \bar\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} = (\gamma^\mu)^\dagger \checkmark$$

b) $(S^{\mu\nu})^\dagger = \gamma^0 S^{\mu\nu}\gamma^0 \qquad (\gamma^0)^2 = \mathbb{1}.$

$$(S^{\mu\nu})^\dagger = -\frac{i}{4}\left[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger\right] = -\frac{i}{4}\left[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0\right]$$

$$= -\frac{i}{4}\gamma^0\left[\gamma^\nu, \gamma^\mu\right]\gamma^0 = \gamma^0 S^{\mu\nu}\gamma^0$$

c) $(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger$

$$= -i\gamma^0\gamma^3\gamma^2\gamma^1$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

## Exercice 6.5

- $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}\mathrm{Tr}(\{\gamma^\mu, \gamma^\nu\}) = \frac{1}{2}\mathrm{Tr}(2\eta^{\mu\nu}\mathbb{1}) = 4\eta^{\mu\nu}$

- $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = \mathrm{Tr}((\{\gamma^\mu, \gamma^\nu\} - \gamma^\nu\gamma^\mu)\gamma^\lambda\gamma^\sigma)$

$$= 2\eta^{\mu\nu}\mathrm{Tr}(\gamma^\lambda\gamma^\sigma) - \mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\sigma)$$

$$= 8\eta^{\mu\nu}\eta^{\lambda\sigma} - \mathrm{Tr}(\gamma^\nu(\{\gamma^\mu, \gamma^\lambda\} - \gamma^\lambda\gamma^\mu)\gamma^\sigma)$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 8\eta^{\mu\nu}\eta^{\lambda\sigma} - 8\eta^{\mu\lambda}\eta^{\nu\sigma} + \mathrm{Tr}(\gamma^\nu\gamma^\lambda\gamma^\mu\gamma^\sigma)$$
$$= 8\eta^{\mu\nu}\eta^{\lambda\sigma} - 8\eta^{\mu\lambda}\eta^{\nu\sigma} + \mathrm{Tr}(\gamma^\nu\gamma^\lambda(\{\gamma^\mu,\gamma^\sigma\} - \gamma^\sigma\gamma^\mu))$$
$$= 8\eta^{\mu\nu}\eta^{\lambda\sigma} - 8\eta^{\mu\lambda}\eta^{\nu\sigma} + 8\eta^{\nu\lambda}\eta^{\mu\sigma} - \mathrm{Tr}(\gamma^\nu\gamma^\lambda\gamma^\sigma\gamma^\mu)$$

$\rightsquigarrow$ $2\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 8(\eta^{\mu\nu}\eta^{\lambda\sigma} - \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\nu\lambda}\eta^{\mu\sigma})$ circularity of the trace

$\rightsquigarrow$ $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\lambda\sigma} - \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\nu\lambda}\eta^{\mu\sigma})$

- $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}) = \mathrm{Tr}(\gamma^5\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}) = \mathrm{Tr}(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma^5)$
$$= -\mathrm{Tr}(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}\gamma^5)$$

$\rightsquigarrow$ $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}) = 0.$

- $\mathrm{Tr}(\gamma^5\gamma^\mu) = \mathrm{Tr}(\gamma^\mu\gamma^5) = -\mathrm{Tr}(\gamma^\mu\gamma^5) \Rightarrow \mathrm{Tr}(\gamma^5\gamma^\mu) = 0$

- Consider $i \neq \mu$ and $i \neq \nu$ Always possible

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^i\gamma^i) \qquad (\gamma^i)^2 = -\mathbb{1}.$$
$$= -\mathrm{Tr}(\gamma^5\gamma^i\gamma^\mu\gamma^\nu\gamma^i)$$
$$= \mathrm{Tr}(\gamma^i\gamma^5\gamma^\mu\gamma^\nu\gamma^i) = -\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu)$$

$\rightsquigarrow$ $\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0.$

- $\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = A^{\mu\nu\lambda\sigma}$

If any two indices are the same $A^{\mu\nu\lambda\sigma} = 0$ as two $\gamma$ make a square leaving an identity matrix such that we are left in one of the previous case.

If all indices are different then the $\gamma$ anticommute and $A$ is a fully antisymmetric tensor. Therefore $A^{\mu\nu\lambda\sigma} = c\,\varepsilon^{\mu\nu\lambda\sigma}$

Moreover $A^{0123} = \mathrm{Tr}(\gamma^5 (-i)\gamma^5) = -i\,\mathrm{Tr}((\gamma^5)^2) = -4i$

Then $\mathrm{Tr}(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = -4i\,\varepsilon^{\mu\nu\lambda\sigma}$

## Exercice 6.6

a) $\mathcal{L} = \bar\psi(i\gamma^\mu \partial_\mu - m)\psi \qquad \psi \to e^{i\alpha}\psi \quad \bar\psi = \psi^\dagger\gamma^0 \to e^{-i\alpha}\psi^\dagger\gamma^0 = e^{-i\alpha}\bar\psi$

$\mathcal{L}' = \bar\psi e^{-i\alpha}(i\gamma^\mu \partial_\mu - m)e^{i\alpha}\psi = \mathcal{L}$ as $\alpha$ constant

b) $J^\mu = \bar\psi\gamma^\mu\psi \qquad \partial_\mu J^\mu = \partial_\mu\bar\psi\,\gamma^\mu\psi + \bar\psi\gamma^\mu\partial_\mu\psi$

Equations of motion $\quad \dfrac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) = 0.$

$\Rightarrow \; -\bar\psi m - \partial_\mu(\bar\psi i\gamma^\mu) = 0$

$\Rightarrow \; m\bar\psi + i\,\partial_\mu\bar\psi\gamma^\mu = 0. \quad \Rightarrow \partial_\mu\bar\psi\gamma^\mu = im\bar\psi$

$\Rightarrow \; m\psi^\dagger\gamma^0 + i\,\partial_\mu\psi^\dagger\gamma^0\gamma^\mu = 0$

$$\rightsquigarrow \quad m\gamma^0\psi - i\left(\gamma^0\gamma^\mu\right)^\dagger \partial_\mu\psi = 0$$

$$\rightsquigarrow \quad m\gamma^0\psi - i\gamma^0\gamma^\mu\partial_\mu\psi = 0$$

$$\rightsquigarrow \quad i\gamma^\mu\partial_\mu\psi = m\psi \quad \text{Dirac's equation}$$

Thus $J^\mu = im\bar\psi\psi - im\bar\psi\psi = 0$.

c) $\mathcal{L} = \bar\psi\, i\gamma^\mu\partial_\mu\psi \qquad \psi \to e^{i\alpha\gamma^5}\psi, \quad \bar\psi = \psi^\dagger\gamma^0 \to \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0$

$$\mathcal{L}' = i\psi^\dagger e^{-i\alpha\gamma^5}\gamma^0\gamma^\mu\partial_\mu e^{i\alpha\gamma^5}\psi$$

$$\mathcal{L}' = i\psi^\dagger e^{-i\alpha\gamma^5}\gamma^0\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu\psi$$

$$= i\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi + i\psi^\dagger\left(e^{-i\alpha\gamma^5}\gamma^0\gamma^\mu - \gamma^0\gamma^\mu e^{-i\alpha\gamma^5}\right)e^{i\alpha\gamma^5}\partial_\mu\psi$$

$$= \mathcal{L} - i\psi^\dagger\left[\gamma^0\gamma^\mu, e^{-i\alpha\gamma^5}\right]e^{i\alpha\gamma^5}\partial_\mu\psi.$$

- $$\left[\gamma^0\gamma^\mu, e^{-i\alpha\gamma^5}\right] = \sum_{n=0}^{+\infty}\frac{(-i\alpha)^n}{n!}\left[\gamma^0\gamma^\mu, (\gamma^5)^n\right]$$

$$= \sum_{p=0}^{+\infty}\frac{(-i\alpha)^{2p}}{2p!}\left[\gamma^0\gamma^\mu, \mathbb{1}\right] + \sum_{p=0}^{+\infty}\frac{(-i\alpha)^{2p+1}}{(2p+1)!}\left[\gamma^0\gamma^\mu, \gamma^5\right]$$

$$\left[\gamma^0\gamma^\mu, \gamma^5\right] = \gamma^0\gamma^\mu\gamma^5 - \gamma^5\gamma^0\gamma^\mu = \gamma^0\gamma^\mu\gamma^5 + \gamma^0\gamma^5\gamma^\mu = 0.$$

$$\rightsquigarrow \left[\gamma^0\gamma^\mu, e^{-i\alpha\gamma^5}\right] = 0 \quad \Rightarrow \mathcal{L}' = \mathcal{L}.$$

d) Dirac's equation $(m=0)$ : $\partial_\mu\bar\psi\gamma^\mu = 0$ and $\gamma^\mu\partial_\mu\psi = 0$

$$\partial_\mu J_5 = \partial_\mu\bar\psi\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi = -\bar\psi\gamma^5\gamma^\mu\partial_\mu\psi = 0.$$

# Exercice 6.7

Plane wave solutions of Dirac's equation:

$$\begin{cases} (i\gamma^\mu\partial_\mu - m)\, u(\vec{p})\, e^{-ipx} = 0 \\ (i\gamma^\mu\partial_\mu - m)\, v(\vec{p})\, e^{+ipx} = 0 \end{cases} \qquad \begin{aligned} e^{-ipx} &= e^{-i\omega t + i p_j x^j} \\ e^{ipx} &= e^{i\omega t - i p_j x^j} \end{aligned}$$

$$\rightsquigarrow \begin{cases} [\gamma^\mu p_\mu - m]\, u(\vec{p}) = 0 \\ [\gamma^\mu p_\mu + m]\, v(\vec{p}) = 0. \end{cases}$$

$$\rightsquigarrow \begin{cases} \begin{pmatrix} -m & \omega + p_j\sigma^j \\ \omega - p_j\sigma^j & -m \end{pmatrix} u(\vec{p}) = 0 \\ \begin{pmatrix} -m & -\omega - p_j\sigma^j \\ -\omega + p_j\sigma^j & -m \end{pmatrix} v(\vec{p}) = 0. \end{cases}$$

Introduce $u(p) = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$, $v(p) = \begin{pmatrix} v_L \\ v_R \end{pmatrix}$

$$\rightsquigarrow \begin{cases} -m u_L + \sigma^\mu p_\mu\, u_R = 0 \\ -m u_R + \bar{\sigma}^\mu p_\mu\, u_L = 0 \end{cases} \quad \text{and} \quad \begin{cases} -m v_L - \sigma^\mu p_\mu\, v_R = 0 \\ -m v_R - \bar{\sigma}^\mu p_\mu\, v_L = 0. \end{cases}$$

$$\rightsquigarrow \begin{cases} u_L = \frac{1}{m}(\sigma^\mu p_\mu) u_R \\ (\bar{\sigma}^\mu p_\mu)(\sigma^\nu p_\nu) u_R - m^2 u_R = 0 \quad (*) \end{cases} \qquad \begin{cases} v_L = -\frac{1}{m}(\sigma^\mu p_\mu) v_R \\ (\bar{\sigma}^\mu p_\mu)(\sigma^\nu p_\nu) v_R - m^2 v_R = 0. \quad (*) \end{cases}$$

$$(\bar{\sigma}^\mu p_\mu)(\sigma^\nu p_\nu) = (\omega - \sigma^i p_i)(\omega + \sigma^j p_j) = \omega^2 - \sigma^i\sigma^j p_i p_j$$

$$= \omega^2 - (\delta^{ij} + i\varepsilon^{ijk}\sigma^k) p_i p_j = \omega^2 - \vec{p}^{\,2} = p^2 = m^2$$

Equations $(*)$, $(*)'$ are readily satisfied on shell. Therefore $u_R$ and $v_R$ are actually unconstrained.

$$\rightsquigarrow \quad u_L = \frac{1}{m}\left(\sigma_\mu p^\mu\right) u_R \quad \text{and} \quad v_L = -\frac{1}{m}\left(\sigma_\mu p^\mu\right) v_R$$

Choose
$$\begin{cases} u_R^i \equiv \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^i & \text{then } u_L^i = \sqrt{\sigma_\mu p^\mu}\, \xi^i \\ v_R^i \equiv -\sqrt{\bar{\sigma}_\mu p^\mu}\, \eta^i & \text{then } v_L^i = \sqrt{\sigma_\mu p^\mu}\, \eta^i \end{cases}$$

The square root makes sense iif the matrices below have real positive eigenvalues. Let us check that.

$$\sigma^\mu p_\mu = \begin{pmatrix} \omega + p_3 & p_1 - i p_2 \\ p_1 + i p_2 & \omega - p_3 \end{pmatrix} \quad \begin{cases} \det\left(\sigma^\mu p_\mu\right) = \omega^2 - p_3^2 - p_1^2 - p_2^2 = m^2 \\ \mathrm{Tr}\left(\sigma^\mu p_\mu\right) = 2\omega > 0 . \end{cases}$$

Let $\lambda_1, \lambda_2$ be the two eigenvalues:

$$\lambda_1 + \lambda_2 = 2\omega \qquad \lambda_1 \lambda_2 = m^2$$

$$\lambda_2 = \frac{m^2}{\lambda_1} \qquad \lambda_1 + \frac{m^2}{\lambda_1} = \omega \qquad \lambda_1^2 - 2\omega\lambda_1 + m^2 = 0$$

$$\Delta = 4\omega^2 - 4m^2 = 4\vec{p}^{\,2}$$

$$\begin{cases} \lambda_1 = \omega \pm |\vec{p}| \\ \lambda_2 = \omega \mp |\vec{p}| \end{cases}$$

Both are positive since $\omega^2 - |\vec{p}|^2 > 0 \Rightarrow \omega > |\vec{p}|$.

# Exerice 6.8

$$\bar{u}^i u^j = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi^i \\ \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^i \end{pmatrix}^\dagger \gamma^0 \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi^j \\ \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^j \end{pmatrix} \qquad (\sigma^\mu)^\dagger = \sigma^\mu$$

$$= \left( (\xi^i)^\dagger \sqrt{\sigma_\mu p^\mu} \quad (\xi^i)^\dagger \sqrt{\bar{\sigma}_\mu p^\mu} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\, \xi^j \\ \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^j \end{pmatrix}$$

$$= (\xi^i)^\dagger \sqrt{\sigma_\mu p^\mu} \sqrt{\bar{\sigma}_\mu p^\mu}\, \xi^j + (\xi^i)^\dagger \sqrt{\bar{\sigma}_\mu p^\mu} \sqrt{\sigma_\mu p^\mu}\, \xi^j$$

$$= 2m (\xi^i)^\dagger \xi^j = 2m\, \delta^{ij}$$

$$\bar{u}^i v^j = \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \\ \sqrt{\bar{\sigma} p}\, \xi^i \end{pmatrix}^\dagger \gamma^0 \begin{pmatrix} \sqrt{\sigma p}\, \eta^j \\ -\sqrt{\bar{\sigma} p}\, \eta^j \end{pmatrix}$$

$$= -(\xi^i)^\dagger \sqrt{\sigma p} \sqrt{\bar{\sigma} p}\, \eta^j + (\xi^i)^\dagger \sqrt{\bar{\sigma} p} \sqrt{\sigma p}\, \eta^j$$

$$= m \left( (\xi^i)^\dagger \eta^j - (\xi^i)^\dagger \eta^j \right) = 0.$$

$$\sum_i u^i \bar{u}^i = \sum_i \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \\ \sqrt{\bar{\sigma} p}\, \xi^i \end{pmatrix} \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \\ \sqrt{\bar{\sigma} p}\, \xi^i \end{pmatrix}^\dagger \gamma^0$$

$$= \sum_i \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \\ \sqrt{\bar{\sigma} p}\, \xi^i \end{pmatrix} \left( (\xi^i)^\dagger \sqrt{\sigma p} \quad (\xi^i)^\dagger \sqrt{\bar{\sigma} p} \right) \gamma^0$$

$$= \sum_i \begin{pmatrix} \sqrt{\sigma p}\, \xi^i \\ \sqrt{\bar{\sigma} p}\, \xi^i \end{pmatrix} \left( (\xi^i)^\dagger \sqrt{\bar{\sigma} p} \quad (\xi^i)^\dagger \sqrt{\sigma p} \right)$$

$$= \begin{pmatrix} \sqrt{\sigma p} \sum_i \xi_i (\xi_i)^\dagger \sqrt{\bar{\sigma} p} & \sqrt{\sigma p} \sum_i \xi_i (\xi_i)^\dagger \sqrt{\sigma p} \\ \sqrt{\bar{\sigma} p} \sum_i \xi_i (\xi_i)^\dagger \sqrt{\bar{\sigma} p} & \sqrt{\bar{\sigma} p} \sum_i \xi_i (\xi_i)^\dagger \sqrt{\sigma p} \end{pmatrix} \qquad \sum_i \xi^i (\xi^i)^\dagger = \mathbb{1}_{2\times 2}$$

$$m \sum_i u^i \bar{u}^i = \begin{pmatrix} m\mathbb{1}_{2\times 2} & \sigma p \\ \bar{\sigma} p & m\mathbb{1}_{2\times 2} \end{pmatrix} = m\mathbb{1}_{4\times 4} + \gamma^\mu p_\mu$$

## Exercice 6.9

Verifier que $\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu_{1\nu} \gamma^\nu$ en utilisant une transformation infinitesimale.

$$\begin{cases} \Lambda_{1/2} = e^{-i/2\, \omega_{\mu\nu} S^{\mu\nu}} \\ \Lambda^\mu_{1\nu} \simeq \delta^\mu_\nu - \frac{i}{2} \omega_{\alpha\beta} \left(J^{\alpha\beta}\right)^\mu{}_\nu \simeq \delta^\mu_\nu + \frac{1}{2} \omega_{\alpha\beta} \left(\eta^{\mu\alpha} \delta^\beta_\nu - \eta^{\mu\beta} \delta^\alpha_\nu\right) + \mathcal{O}(\omega^2) \end{cases}$$

$$\Lambda_{1/2} \simeq \mathbb{1} - \frac{i}{2} \omega_{\alpha\beta} S^{\alpha\beta} + \mathcal{O}(\omega^2)$$

$$\Lambda^{-1}_{1/2} \simeq \mathbb{1} + \frac{i}{2} \omega_{\rho\sigma} S^{\rho\sigma} + \mathcal{O}(\omega^2)$$

Thus
$$\begin{aligned}
\Lambda^{-1}_{1/2} \gamma^\mu \Lambda_{1/2} &\simeq \left(\mathbb{1} + \frac{i}{2} \omega_{\alpha\beta} S^{\alpha\beta}\right) \gamma^\mu \left(\mathbb{1} - \frac{i}{2} \omega_{\rho\sigma} S^{\rho\sigma}\right) + \mathcal{O}(\omega^2) \\
&\simeq \gamma^\mu - \frac{i}{2} \omega_{\alpha\beta} \left[\gamma^\mu, S^{\alpha\beta}\right] + \mathcal{O}(\omega^2) \\
&\simeq \gamma^\mu - \frac{i}{2} \omega_{\alpha\beta} \frac{i}{4} \left[\gamma^\mu, \left[\gamma^\alpha, \gamma^\beta\right]\right] + \mathcal{O}(\omega^2) \\
&\simeq \gamma^\mu + \frac{1}{8} \omega_{\alpha\beta} (-4) \left(\eta^{\mu\beta} \gamma^\alpha - \eta^{\mu\alpha} \gamma^\beta\right) + \mathcal{O}(\omega^2) \\
&\simeq \delta^\mu_\nu \gamma^\nu + \frac{1}{2} \omega_{\alpha\beta} \left(\eta^{\mu\alpha} \delta^\beta_\nu - \eta^{\mu\beta} \delta^\alpha_\nu\right) \gamma^\nu + \mathcal{O}(\omega^2) \\
&\simeq \Lambda^\mu_{1\nu} \gamma^\nu + \mathcal{O}(\omega^2)
\end{aligned}$$

# Exercice 6.10

a) $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ and $\psi' = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi$

Moreover $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix}\right]$

$$= \frac{i}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$

$\leadsto \psi'_{L/R} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}_{L/R}}\psi_{L/R}$

with $\begin{cases} S_L^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right) \\ S_R^{\mu\nu} = \frac{i}{4}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right) \end{cases}$

$$\begin{cases} S_L^{ij} = \frac{i}{4}\left(\sigma^j\sigma^i - \sigma^i\sigma^j\right) = \frac{i}{4}[\sigma^j, \sigma^i] = \frac{i}{4}2i\,\varepsilon^{jik}\sigma^k = \frac{1}{2}\varepsilon^{ijk}\sigma^k \\ S_R^{ij} = S_L^{ij}. \end{cases}$$

$$\begin{cases} S_L^{0i} = \frac{i}{4}\left(-\sigma^0\sigma^i - \sigma^i\sigma^0\right) = -\frac{1}{2}i\sigma^i \\ S_R^{0i} = \frac{i}{4}\left(\sigma^0\sigma^i + \sigma^i\sigma^0\right) = \frac{1}{2}i\sigma^i = -S_L^{0i} \end{cases}$$

Dirac's representation is reducible: $\underset{su(2)\oplus su(2)}{\left(\frac{1}{2}, 0\right)} \oplus \underset{su(2)\oplus su(2)}{\left(0, \frac{1}{2}\right)}$

Both representations $(1/2, 0)$ and $(0, 1/2)$ act the same on rotations but they differ on boosts.

c) Let us denote $\chi = \sigma^2 \psi_R^*$

$$\psi_R' = e^{-i/2\, \omega_{\mu\nu} S_R^{\mu\nu}} \psi_R \quad \leadsto \quad \psi_R'^* = e^{i/2\, \omega_{\mu\nu} (S_R^{\mu\nu})^*} \psi_R^*$$

Thus $\chi' = \sigma^2 e^{i/2\, \omega_{\mu\nu} (S_R^{\mu\nu})^*} \psi_R^*$

$$\begin{aligned} \omega_{\mu\nu} S_R^{\mu\nu} &= \omega_{ij} S_R^{ij} + 2\omega_{0i} S_R^{0i} \\ &= \omega_{ij} \frac{1}{2} \varepsilon^{ijk} \sigma^k + i\omega_{0j} \sigma^j \\ &= \left( \frac{1}{2} \omega_{ij} \varepsilon^{ijk} + i\omega_{0k} \right) \sigma^k \end{aligned}$$

Thus $\omega_{\mu\nu} (S_R^{\mu\nu})^* = \left( \frac{1}{2} \omega_{ij} \varepsilon^{ijk} - i\omega_{0k} \right) (\sigma^k)^*$

And $$\begin{aligned} \sigma^2 \omega_{\mu\nu} (S_R^{\mu\nu})^* &= -\left( \frac{1}{2} \omega_{ij} \varepsilon^{ijk} - i\omega_{0k} \right) \sigma^k \sigma^2 \\ &= \left( -\omega_{ij} S_R^{ij} + 2\omega_{0i} S_R^{0i} \right) \sigma^2 \\ &= -\omega_{\mu\nu} S_L^{\mu\nu} \sigma^2 \end{aligned}$$

Therefore: $\sigma^2 \left[ \frac{i}{2} \omega_{\mu\nu} (S_R^{\mu\nu})^* \right] = \left[ -\frac{i}{2} \omega_{\mu\nu} S_L^{\mu\nu} \right] \sigma^2$

And $\sigma^2 \left[ \frac{i}{2} \omega_{\mu\nu} (S_R^{\mu\nu})^* \right]^n = \left[ -\frac{i}{2} \omega_{\mu\nu} S_L^{\mu\nu} \right]^n \sigma^2$

Thus $$\sigma^2 e^{i/2\, \omega_{\mu\nu} (S_R^{\mu\nu})^*} = e^{-i/2\, \omega_{\mu\nu} S_L^{\mu\nu}} \sigma^2$$

In the end: $\chi' = e^{-i/2\, \omega_{\mu\nu} S_L^{\mu\nu}} \chi$ $\Rightarrow$ Transform as a left handed spinor.

Same thing for $\sigma^2 \psi_L^*$ which transforms as a right handed spinor.

d) $i\gamma^\mu \partial_\mu \psi = m\psi$

$$\rightsquigarrow \quad i\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \partial_\mu \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = m\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\rightsquigarrow \begin{cases} i\sigma^\mu \partial_\mu \psi_R = m\psi_L \\ i\bar{\sigma}^\mu \partial_\mu \psi_L = m\psi_R \end{cases}$$

When $m=0$ both $\psi_L$ and $\psi_R$ are independant.

g) $m\bar{\psi}\psi = m(\psi_R^\dagger \psi_L + \psi_L^\dagger \psi_R)$

But we could also define "$\psi_L = \sigma^2 \psi_R^*$"

In which case $m\left(\psi_R^\dagger \sigma^2 \psi_R^* + \psi_R^T \sigma^2 \psi_R\right)$

$m\left(\psi_R^\dagger \sigma^2 \psi_R^* + (\psi_R^\dagger \sigma^2 \psi_R^*)^\dagger\right)$

$m\left(\psi_R^\dagger \sigma^2 \psi_R^* + h.c\right)$

## Exercice 7.1

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip(x-y)}$$

$$(i\gamma^\mu\partial_\mu - m) S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)(\not{p}-m)}{p^2-m^2+i\varepsilon} e^{-ip(x-y)}$$

$$= i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}$$

$$= i\delta^{(4)}(x-y)$$

## Exercice 7.2

$$J_\mu = \bar{\Psi}\gamma^\mu\psi \quad \leadsto \quad Q = \int d^3\vec{x}\; J_0 = \int d^3\vec{x}\; \bar{\Psi}\gamma^0\psi$$

$$= \int d^3\vec{x}\; \psi^\dagger\psi .$$

$$\psi = \sum_{i=1}^{2} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left( a_k^i u^i(\vec{k}) e^{-ikx} + b_k^{i\dagger} v^i(\vec{k}) e^{ikx} \right)$$

$$\psi^\dagger = \sum_{j=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \left( a_p^{j\dagger} u^{j\dagger}(\vec{p}) e^{ipx} + b_p^j v^{j\dagger}(\vec{p}) e^{-ipx} \right)$$

$$\leadsto Q = \sum_{\substack{i=1\\j=1}}^{2} \int \frac{d^3\vec{p}\, d^3\vec{k}}{(2\pi)^3 (2\omega_k\omega_p)} \int d^3\vec{x} \Big\{ a_p^{j\dagger} a_k^i\, u^{j\dagger}(\vec{p}) u^i(\vec{k})\, e^{i(p-k)x}$$

$$+ a_p^{j\dagger} b_k^{i\dagger}\, u^{j\dagger}(\vec{p}) v^i(\vec{k})\, e^{i(p+k)x}$$

$$+ b_p^j a_k^i\, v^{j\dagger}(\vec{p}) u^i(\vec{k})\, e^{-i(p+k)x}$$

$$+ b_p^j b_k^{i\dagger}\, v^{j\dagger}(\vec{p}) v^i(\vec{k})\, e^{-i(p-k)x} \Big\} .$$

$$Q = \sum_{j=1}^{2} \int \frac{d^3\vec{p}}{2\omega_p} \Big\{ a_p^{j\dagger} a_p^{i} \, u^{j\dagger}(\vec{p}) u^{i}(\vec{p}) + a_p^{j\dagger} b_{-p}^{i\dagger} \, u^{j\dagger}(\vec{p}) v^{i}(-\vec{p}) \, e^{2i\omega_p t}$$

$$b_p^{j} a_{-p}^{i} \, v^{j\dagger}(\vec{p}) u^{i}(-\vec{p}) \, e^{-2i\omega_p t} + b_p^{j} b_p^{i\dagger} \, v^{j\dagger}(\vec{p}) v^{i}(\vec{p}) \Big\}$$

$$Q = \int d^3\vec{p} \, \Big\{ a_p^{i\dagger} a_p + b_p^{i} b_p^{i\dagger} \Big\}$$